The deformation near the point $r = e_j$ is given by

$$\delta_j = |b_j| V_j \quad (24)$$

where b_j is defined by Eq (9e) Continuity of the deformation requires that

$$\sum_{j=1\ 2}^{n} \delta_j = 0 \quad (25)$$

The solution of Eqs (23–25) is

$$V = -\pi p \sum_{j=1}^{n} |b_j| e_j^2 \Big/ \sum_{j=1}^{n} |b_j| \quad (26)$$

As before, V, which gives the membrane stresses [Eq (1)] is easily computed The quantities V_j, which give the bending stresses [Eqs (9a–9d)] at each point $r = e_j$, are easily obtained from Eq (23)

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Large-Amplitude Vibration and Response of Curved Panels

Benjamin E Cummings*

Univei sity of Califoi nia, Los Angeles, Calif

The dynamic nonlinear shallow-shell equations are examined in the special case of a cylindrical shell segment Different developments are given for two systems: in system A, the stress boundary conditions aie satisfied exactly, and compatibility is satisfied on the average; in system B, compatibility is satisfied exactly, and the stress boundary conditions are satisfied on the average Perturbation and exact integral expressions are found for the frequencies of vibration The response to delta-function, step-function, and haimonic-function loading is examined Dynamic buckling is predicted by shock response method

Nomenclature

A,B	=	time dependent amplitudes, displacement and stress functions
D	=	bending stiffness $= Eh^3/12(1 - \nu^2)$
E	=	Young's modulus
F	=	stress function
G	=	energy parameter [Eq (16)]; greater definition of all parameters may be found in Ref 4
K	=	energy constant [Eq (15)]
L	=	length of the panel
N_x,N_y,N_{xi},N_{yi}	=	stress resultants
P	=	pressure loading
Q	=	generalized force
R	=	step function amplitude
R_Δ	=	response ratio
S	=	relative displacement of the panel
T	=	period of vibration
AR	=	aspect ratio
a	=	cylinder radius
g	=	nonlinearity parameter
h	=	panel thickness
m	=	mass per unit area
n	=	circumfeiential mode number
p	=	ψ_τ
q	=	reduced nondimensional forcing function
r	=	reduced step amplitude
t	=	time variable
w	=	panel displacement
x,y,z	=	space variables
α	=	reduced amplitude parameter
β	=	energy parameter [Eq (24)]
γ	=	frequency parameter [Eq (A7)]
δ	=	perturbation parameter
Δ	=	displacement parameter
ϵ	=	nonlinearity parameter
θ	=	special form of ψ
χ	=	nonlinearity parameter
η,μ	=	nonlinearity parameter
τ	=	nondimensional time
ϕ	=	special form of ψ
ψ	=	nondimensional amplitude
ω	=	frequency
Ω	=	nondimensional frequency

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* Assistant Professor of Engineering; formerly Member of the Senior Technical Staff, Institute for Defense Analysis, Washington D C

Introduction

Two solutions to the shallow-shell equations are considered in this paper The first satisfies the stress boundary conditions exactly, but (in accordance with the Galerkin method) it satisfies compatibility only on the average This development is referred to as system A The second solution

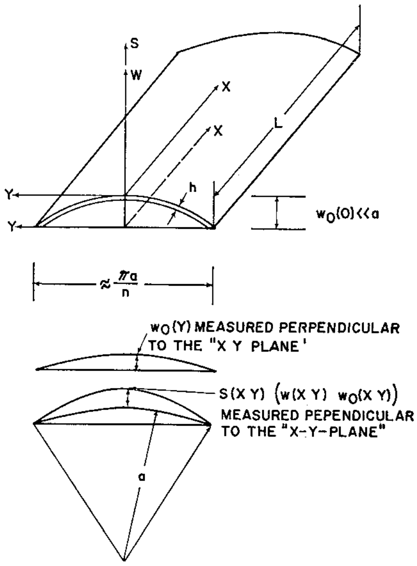

Fig 1 Panel geometry

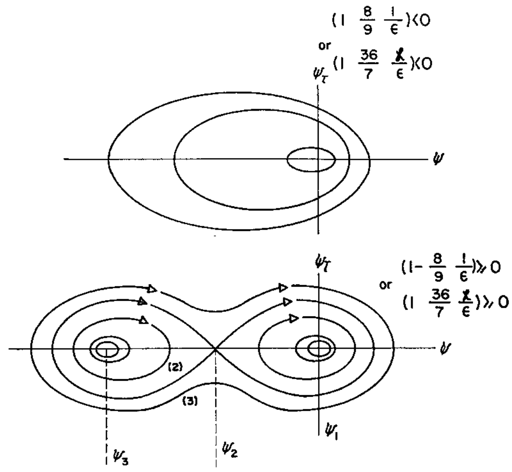

Fig 2 Effect of nonlinearity parameters on phase-plane contours

satisfies compatibility exactly, but it satisfies the stress boundary conditions only on the average This development is referred to as system B System B is believed to represent the phenomena encountered in the physical world better, but the results have not been compared with physical experiments System A is a less complex way of studying the main characteristics of the problem than system B The format of this paper facilitates comparisons of the two systems: Whenever the systems differ significantly, the page is divided and the contrasting items are presented side by side Suffixes "a" and "b" in equation numbers indicate to which system a given equation belongs Wherever the page is divided, system A always appears on the left Detailed development of the perturbation solution in a form that applies to both systems appears in the Appendix

Basic Equations

The basic equations from which this analysis begins are the equations of a curved plate (the curved-plate geometry is shown in Fig 1) or shallow shell They are derived from the work of Marguerre or von Karman [1-4] The forms used here are

$$D\nabla^4 S = P(x,y,t) - S_{tt}m + N_{xi}S_{xx} + N_{yi}S_{yy} - (1/a)F_{xx} + [F_{xx}S_{yy} + F_{yy}S_{xx} - 2F_{xy}S_{xy}] \quad (1)$$

$$(1/Eh)\nabla^4 F = (1/a)S_{xx} + [S_{xy}^2 - S_{xx}S_{yy}] \quad (2)$$

The boundary conditions are the so-called freely supported conditions:

$$\left.\begin{matrix} S = S_{xx} = 0 \\ F = F_{xx} = 0 \end{matrix}\right\} \text{ on } x = L,O \quad (3)$$

$$\left.\begin{matrix} S = S_{yy} = 0 \\ F = F_{yy} = 0 \end{matrix}\right\} \text{ on } y = \pm(\pi a/2n) \quad (4)$$

Assume displacement and stress function modes of the form that satisfy the linearized form of Eqs (1) and (2):

$$S = A(t)\cos(ny/a)\sin(\pi x/L) \quad (5a)$$

$$F = B(t)\cos(ny/a)\sin(\pi x/L) \quad (6a)$$

These equations satisfy the boundary conditions of Eqs (3) and (4) identically By introducing Eqs (5a) and (6a) into Eqs (1) and (2), and applying to a Galerkin averaging technique, both the equilibrium and compatibility conditions are satisfied on the average

The resulting relations are

$$A_{tt} + A\left\{D\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 + N_{xi}\left(\frac{\pi}{L}\right)^2 + N_{yi}\left(\frac{n}{a}\right)^2\right\} - \frac{1}{a}\left(\frac{\pi}{L}\right)^2 B^2 - \frac{32}{3\pi}\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^2 AB = \left(\frac{n}{a}\right)\left(\frac{4}{L\pi}\right)P(t)$$

$$B = -\frac{Eh\,[\frac{1}{3}A^2(4n/aL)^2 + (A/a)(\pi/L)^2]}{[(\pi/L)^2 + (n/a)^2]^2}$$

If the coefficient B is eliminated from these two equations, and the governing differential equation for A is written in terms of dimensionless variables,

$$\psi_{\tau\tau} + \Omega_L^2[\psi + \epsilon(\psi^2 + \tfrac{2}{9}\psi^3)] = Q(\tau) \quad (7a)$$

Equation (7a) has also been obtained by a different method (Ref 1)

Assume a displacement mode that corresponds to displacement solution of the linearized form of Eqs (1) and (2):

$$S = A(t)\cos(ny/a)\sin(\pi x/L) \quad (5b)$$

When Eq (5b) is substituted into Eq (2), it is found that

$$F = A(t)Eh\left\{-\frac{1}{a}\left(\frac{\pi}{L}\right)^2\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^{-2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{ny}{a}\right) + A(t)\left[\left(\frac{n}{a}\right)^2\left(\frac{\pi}{L}\right)^{-2}\cos\left(\frac{2\pi x}{L}\right) - \left(\frac{\pi}{L}\right)^2\left(\frac{n}{a}\right)^{-2}\cos\left(\frac{2ny}{a}\right)\right]\right\} \quad (6b)$$

If S is given by Eq (5b), then F, as given by Eq (6b), satisfies compatibility exactly and Eqs (3) and (4) on the average

By substituting Eqs (5b) and (6b) into Eq (1) and applying a Galerkin averaging technique, the equilibrium condition is satisfied on the average The resulting equation written in terms of nondimensional variables is

$$\psi_{\tau\tau} + \Omega_L^2[\psi + (7\epsilon/9)(\psi^2 + \chi\psi^3)] = Q(\tau) \quad (7b)$$

where

$$\chi = \tfrac{9}{7}\,(\tfrac{1}{4})(\pi/4)^4(AR)^{-4}(1 + AR^2)^2(1 + AR^4)$$

$$AR = nL/\pi a$$

Equation (7b) is similar to Eq (7a) in form but never yields exactly the same results

Free Vibration

For free vibration, the right-hand side of Eq (7) is set equal to zero, from which

$$\psi_{\tau\tau} + \Omega_L^2 g(\psi) = 0 \tag{8}$$

where

$$g(\psi) = \psi + \epsilon\,(\psi^2 + \tfrac{2}{9}\,\psi^3) \tag{9a}$$

$$g(\psi) = \psi + (7\epsilon/9)\,(\psi^2 + \chi\,\psi^3) \tag{9b}$$

Equation (8) may be written in the form

$$p\psi = -\Omega_L^2 g(\psi)/p \tag{10}$$

where $p \equiv \psi_\tau$ Equation (10) is known to have singularities when

$$g(\psi) = p = 0 \tag{11}$$

These singularities are at the points $(\psi_1,0)$, $(\psi_2,0)$, and $(\psi_3,0)$ in the phase plane where

$$\begin{aligned} \psi_1 &= 0 \\ \psi_2 &= -\frac{9}{4}\left[1 - \left(1 - \frac{8}{9}\,\frac{1}{\epsilon}\right)^{1/2}\right] \\ \psi_3 &= -\frac{9}{4}\left[1 + \left(1 - \frac{8}{9}\,\frac{1}{\epsilon}\right)^{1/2}\right] \end{aligned} \tag{12a}$$

$$\begin{aligned} \psi_1 &= 0 \\ \psi_2 &= -\frac{1}{2\chi}\left[1 - \left(1 - \frac{36}{7}\,\frac{\chi}{\epsilon}\right)^{1/2}\right] \\ \psi_3 &= -\frac{1}{2\chi}\left[1 + \left(1 - \frac{36}{7}\,\frac{\chi}{\epsilon}\right)^{1/2}\right] \end{aligned} \tag{12b}$$

If ψ_2 and ψ_3 are complex, then ψ_1 is a stable center and the only singularity in the phase plane If ψ_2 and ψ_3 are real, that is, if

$$[1 - \tfrac{8}{9}\,(1/\epsilon)] > 0 \tag{13a}$$

$$[1 - \tfrac{36}{7}(\chi/\epsilon)] > 0 \tag{13b}$$

then ψ_1 and ψ_3 are stable centers, and ψ_2 is a saddle point Physically, three types of motion are possible under the conditions of Eq (13) The first is limited-amplitude motion about the initial equilibrium position The second is a limited-amplitude motion about the post-buckled equilibrium position The third possibility is a motion of much larger amplitude enclosing both the initial and post-buckled equilibrium points ψ_1 and ψ_τ (see Fig 2)

If Eq (8) is multiplied by ψ_τ and integrated,

$$-\int \psi_\tau d(\psi_\tau) = \Omega_L^2 \int g(\psi) d\psi \tag{14}$$

or

$$(\psi_\tau)^2 = 2K - G(\psi) \tag{15}$$

where

$$G(\psi) = \Omega_L^2[\psi^2 + \epsilon(\tfrac{2}{3}\psi^3 + \tfrac{1}{9}\psi^4)] \tag{16a}$$

$$G(\psi) = \Omega_L^2\{\psi^2 + (7\epsilon/9)[\tfrac{2}{3}\psi^3 + (\chi/2)\psi^4]\} \tag{16b}$$

If the definition of τ is restricted to the special case where

$$\tau = \tau' = \omega_L t \tag{17}$$

and the frequency ω_L is replaced with its associated period T_L, then $\tau' = 2\pi t/T_L$ and $\Omega_L^2 = 1$ Inversion of ψ_τ and integration over a half cycle yields the expression for the period of motion:

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi}\int_{\psi a}^{\psi b} [2K - G(\psi)]^{-1/2}\, d\psi \tag{18}$$

Here T_{NL} is the period of traversal of one complete phase-plane trajectory The value obtained in Eq (18) is unity if $\epsilon = 0$ Equation (18) is an elliptic integral, which is obtained in standard form after finding the roots of

$$2K - G(\psi) = 0 \tag{19}$$

and using them in a transformation of Eq (18) into standard form in accordance with Appendix I of Ref 4 This solution is easy with the use of a digital computer but not easy when carried out by hand Reissner[1] used the Lindstedt-Duffing perturbation technique to find the approximation period of Eq (7a) in a form for simple hand calculation The same technique has been applied to Eq (8) (see Appendix) The results of these solutions are

$$\frac{T_{NL}}{T_L} = \left[1 + \frac{\epsilon}{6}\,(1 - 5\epsilon)\alpha^2\right]^{-1/2} \tag{20a}$$

$$\frac{T_{NL}}{T_L} = \left[1 + \frac{7\epsilon}{9}\left(\frac{3\chi}{4} - \frac{35\epsilon}{54}\right)\alpha^2\right]^{-1/2} \tag{20b}$$

Reissner pointed out the interesting result that, because of the negative sign in Eq (20a), the period of the large-amplitude vibration can be either greater or less than that of the associated linear system, depending on whether $\epsilon < \frac{1}{5}$ or $\epsilon > \frac{1}{5}$

A similar condition applies to Eq (20b) with the dividing line at $\epsilon/\chi = \frac{81}{70}$ Figure 3 shows a comparison of the periods computed from Eqs (16a, 18, and 20a) In the example of a curved aluminum panel with the characteristics $\epsilon = 0\ 10$, $a = 40$ in , $w_0 = 1\ 96$ in , $n = 10$, $A_0 = 0\ 987$ in , $\alpha = 4$, $L = 80$ in panel width $= 12\ 5$ in , $h = 0\ 040$ in , and internal pressure $= 0\ 427$ psi, the exact expressions for the period of vibration by perturbation and integral expressions differs by only 3% In this case, the perturbation results seem preferable to the more time-consuming exact integral method

Response Problems

There is a special class of transient response problems in which the knowledge sought is the maximum deflection amplitude achieved from a single pulse loading For this class of problems, the use of phase-plane trajectories can provide a short cut to the

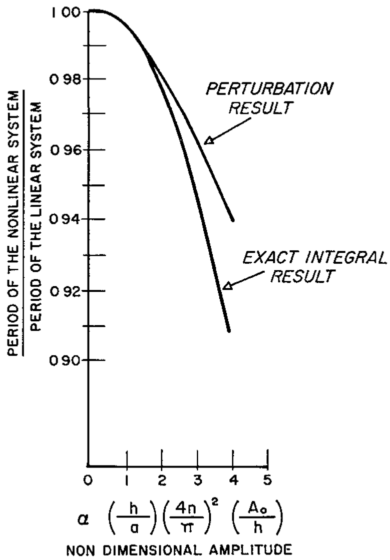

Fig 3 Comparison of perturbation and exact integral frequencies with $\epsilon = 0\ 1$ in system A

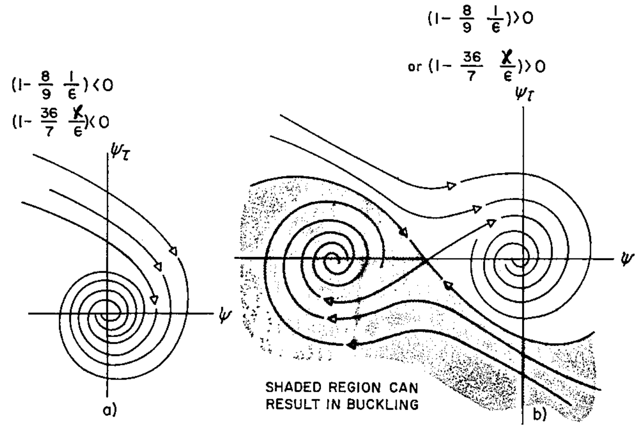

Fig 4 Effect of damping on phase-plane contours

answer Numerical integration of the equation of motion over the duration of the transient loading will yield a phase-plane point (ψ, ψ_τ) which may be used to determine a phase-plane trajectory Once this phase-plane trajectory is known, its maximum displacement is established This method depends on the loading amplitude returning to zero at the end of the loading integral (no step or ramp functions)

The simplest example is for an idealized delta-function loading The integral over the loading identifies an initial velocity $\psi_{\tau 0}$ at displacement Then from Eq (15), assuming Eq (17), the maximum displacement is found as a root of

$$\psi_{\tau 0}{}^2 - \left[\psi^2 + \epsilon\left(\frac{2}{3}\psi^3 + \frac{1}{9}\psi^4\right)\right] = 0 \qquad (21a)$$

$$\psi_{\tau 0}{}^2 - \left[\psi^2 + \frac{7\epsilon}{9}\left(\frac{2}{3}\psi^3 + \frac{\chi}{2}\psi^4\right)\right] = 0 \qquad (21b)$$

The inverse problem is to find the magnitude of pressure applied as a delta function which will cause buckling of the shell If the loading is given by

$$P(x,y,t) = P_0 P^*(x,y)\delta(t) \qquad (22)$$

where

$$|-1 \leq P^*(x,y) \leq +1$$

then the critical value of P_0 is given by

$$P = \frac{\beta(\pi/4n)\ ^3 a^2 m\omega_L L}{\int_0^L \int_{-\pi a/n}^{\pi a/n} P^*(x,y)\cos\left(\frac{ny}{a}\right)\sin\left(\frac{\pi x}{L}\right)dxdy} \qquad (23)$$

where

$$\beta = \left[\psi_2{}^2 + \epsilon\left(\frac{2}{3}\psi_2{}^3 + \frac{1}{9}\psi_2{}^4\right)\right]^{1/2} \qquad (24a)$$

$$\beta = \left[\psi_2{}^2 + \frac{7\epsilon}{9}\left(\frac{2}{3}\psi_2{}^3 + \frac{\chi}{2}\psi_2{}^4\right)\right]^{1/2} \qquad (24b)$$

The effect of including a small amount of damping on the response is indicated by phase-plane trajectories illustrated in Figs 4a and 4b Figure 4b shows that, when damping is present, buckling is caused only for combinations of ψ_τ and ψ in the shaded region Furthermore, an internal pressure pulse will cause buckling only when the initial deflection is already very nearly that of the buckled state †

The response of the curved panel to a loading that is a step function of time can be handled by a coordinate transformation Consider the equation

$$\psi_{\tau\tau} + \Omega_L{}^2 g(\psi) = R\ 1(t) \qquad (25)$$

with initial conditions

$$\psi(0) = \psi_\tau(0) = 0$$

Now let

$$\psi = \phi + \theta(\tau) \qquad (26)$$

where ϕ is a constant such that

$$g(\phi) = r = R/\Omega_L{}^2 \qquad (27)$$

Then Eq (25) becomes‡

$$\theta_{\tau\tau} + \Omega_L{}^2[g'(\phi)\theta + \tfrac{1}{2}g''(\phi)\theta^2 + \tfrac{1}{6}g'''(\phi)\theta^3] = 0 \qquad (28)$$

† See Fig 5 for the effect of pressure on the energy curves and the location of stable equilibrium

‡ In Eq (28) and elsewhere, primes indicate differentiation with respect to the argument The number of primes corresponds to the order of one derivative

with the initial conditions $\theta(0) = -\phi$ and $\theta_\tau(0) = 0$

The associated expression for the period of vibration is

$$\frac{T_{NL}}{T_L} = \frac{1}{\pi}\int_{\theta a}^{\theta b}\left\{2K - \left[g'(\phi)\theta^2 + \frac{1}{2}g''(\phi)\theta^3 + \frac{1}{6}g'''(\phi)\theta^4\right]\right\}^{-1/2} d\theta \qquad (29)$$

where θ_a and θ_b are the real roots of

$$g'(\phi)\theta^2 + \tfrac{1}{2}g''(\phi)\theta^3 + \tfrac{1}{6}g'''(\phi)\theta^4 - 2K = 0 \qquad (30)$$

The perturbation expression for the period of the vibration caused by the step function is

$$\frac{T_{NL}}{T_L} = \left[1 + \phi^2\left\{\frac{2}{3}\left(\frac{g''(\phi)}{g'(\phi)}\right) - \frac{1}{8}\left(\frac{g'''(\phi)}{g'(\phi)}\right)\right\}\right]^{-1/2} \qquad (31)$$

Shock Response Method

The shock response methods of Fung and Barton[5] may be applied to Eq (25) to find the magnitude of r at which a step-function loading will cause dynamic buckling The response ratio R_Δ is defined as the ratio of the maximum response of a nonlinear system to the maximum response of a linear system ($\epsilon = 0$) when each is subjected to the same loading It is shown in Ref 5 that, for step functions, the response ratio is given by

$$R_\Delta = \left\{1 + 2\epsilon\int_0^{\psi 0*} g(\psi^*)d\psi^*\right\}^{-1} \qquad (32)$$

where $\psi^* = \psi/\Delta$ If the buckling condition is desired, the upper limit is -1 and Δ is a deflection corresponding to ψ_1 Evaluation of Eq (32) yields

$$R_\Delta = [1 - (5\epsilon/18)]^{-1} \qquad (33a)$$

$$R_\Delta = \{1 + (14\epsilon/9)[(\chi/4) - \tfrac{1}{3}]\}^{-1} \qquad (33b)$$

Since $|2r|$ is the maximum linear-system response,

$$\Delta = \frac{|2r|}{[1 - (5\epsilon/18)]} \qquad (34a)$$

$$\Delta = \frac{|2r|}{1 + (7\epsilon/54)(3\chi - 4)} \qquad (34b)$$

An effect of an internal or external pressure step is to shift the energy curve associated with the equations of motion as illustrated by the zero-velocity contour shown in Fig 6 This shift is accounted for by evaluating the energy integral, starting with Eq (25) The resultant equation for phase-plane contours is

$$\psi_\tau^2 + \Omega_L^2 G(\psi) = 2K \qquad (35)$$

where $G(\psi)$ is given by Eq (16) The saddle point ψ_1 is obtained as the zero-slope condition of Eq (35) taken with $\psi_\tau = 0$, that is, by

$$\epsilon\left[\tfrac{4}{9}\psi^3 + 2\psi^2\right] + 2\psi - 2r = 0 \qquad (36a)$$

$$(14\chi\epsilon/9)\psi^3 + (14\epsilon/9)\psi^2 + 2\psi - 2r = 0 \qquad (36b)$$

For system A, this information predicts a value of r at which dynamic buckling occurs (see Fig 7) Comparison of such predictions with numerical integration of Eq (7a), when subjected to a step function, showed agreement to three significant figures

Because of the more complex relations in the development of system B, two relations are presented in Fig 8 The first is the minimum value of ϵ for which Eq (13a) is valid as a function of χ If Eq (13a) is not valid, buckling is not possible To each of

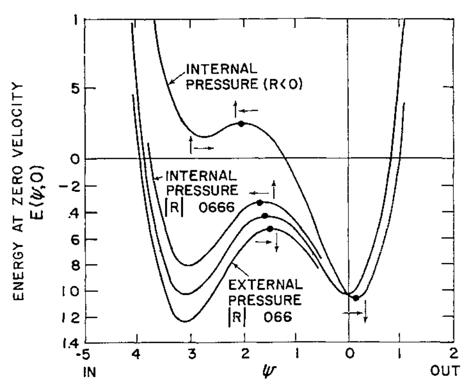

Fig 5 Influence of pressure on energy profiles

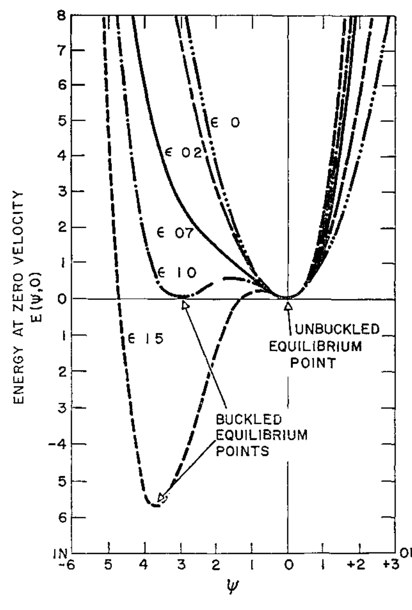

Fig 6 Effect of nonlinearity parameter ($\epsilon \rightarrow$ large as initial stress $\rightarrow$ Buckling) on energy profiles of system A

these minimum values of ϵ there corresponds a value of r which will cause dynamic buckling But at any particular value of χ there may be $\epsilon > \epsilon_{min}$ The value of r which will cause buckling for the larger ϵ, at a given value of χ, decreases as ϵ increases from ϵ_{min} As a result, the value of r which corresponds to ϵ_{min} is designated r_{max}

Periodic Solution

The analysis of forced vibrations by the Lindstedt-Duffing method has been carried out in detail in the Appendix The results show that it is possible to find perturbation, i e , small-amplitude periodic solutions for ψ in the neighborhood of the singular points in the phase-plane as described in Eq (12) Of course, the perturbation associated with the saddle point ψ_2 is not stable and, consequently, not of importance

The equations for the dependence of frequency on amplitude of response are

$$\Omega^2 = 1 + \alpha^2\left(\frac{\epsilon}{\gamma^2}\right)\left[\frac{2}{3}\left(\frac{1}{4} - \psi_0\,\delta\right) - \left(\frac{3-8\gamma^2}{2-8\gamma^2}\right)\delta\right] - \frac{q}{\gamma^2\alpha}$$

$$\gamma^2 = 1 + \epsilon[2\psi_0 + \tfrac{2}{3}\psi_0{}^2]$$

$$\delta = \epsilon[1 + \tfrac{2}{3}\psi_0] \qquad (37a)$$

and $\psi_0 = 0$, or

$$\psi_0 = \psi_3 = -\frac{9}{4}\left[1 + \left(1 - \frac{8}{9}\frac{1}{\epsilon}\right)^{1/2}\right]$$

$$\Omega^2 = 1 + \alpha^2\left(\frac{7\epsilon}{9\gamma^2}\right)\left[3\chi\left(\frac{1}{4} - \psi_0\delta\right) - \left(\frac{3-8\gamma^2}{2-8\gamma^2}\right)\delta\right] - \frac{q}{\gamma^2\alpha}$$

$$\gamma^2 = 1 + (7\epsilon/9)[2\psi_0 + 3\chi\psi_0{}^2]$$

$$\delta = (7\epsilon/9)[1 + (7\epsilon/3)\psi_0] \qquad (37b)$$

and $\psi_0 = 0$, or

$$\psi_0 = \psi_3 = -\frac{1}{2\chi}\left[1 + \left(1 - \frac{36\chi}{7\epsilon}\right)^{1/2}\right]$$

In the particular case where the motion studied starts near the undeflected equilibrium position, the frequency is given by

$$\Omega^2 = 1 + \tfrac{1}{6}\alpha^2\epsilon(1 - 5\epsilon) - q/\alpha \qquad (38a)$$

$$\Omega^2 = 1 + \alpha^2(7\epsilon/9)[(3\chi/4) - (35\epsilon/54)] - q/\alpha \qquad (38b)$$

In Eqs (37) and (38), the periodic forcing function has a small amplitude on the order of α^3 If the amplitude of the forcing function is small (of the order of α^4), terms of the order of α^3 can be added to the frequency equation But since the condition that $\alpha \ll 1$ holds, this higher-order term will be important only when

$$\epsilon \approx \tfrac{1}{5} \qquad (39a)$$

$$\epsilon/\chi \approx \tfrac{81}{70} \qquad (39b)$$

The more interesting case is that which starts near the buckled equilibrium position, where

$$\psi_0 = -\frac{9}{4}\left[1 + \left(1 - \frac{8}{9}\frac{1}{\epsilon}\right)^{1/2}\right] \qquad \psi_0 = -\frac{1}{2\chi}\left[1 + \left(1 - \frac{36\chi}{7\epsilon}\right)^{1/2}\right]$$

For the sake of simplicity, only the first development is discussed here The frequency equation for the undeflected and buckled equilibrium conditions are compared as functions of ϵ in Table 1 (see Fig 6 for the corresponding energy curves)

Table 1 reveals two trends The first is that buckled equilibrium approaches zero frequency more slowly with increasing ϵ (that is, with initial stresses approaching the buckling stress) than does the undeflected equilibrium The second is that the response of the buckled equilibrium becomes much more sharply tuned than that of the undeflected equilibrium with increasing ϵ This effect is represented by the division of (q/α) by the coefficient γ^2, which is always unity at the undeflected equilibruim but which grows with increasing ϵ at the buckled equilibrium

The negative coefficient of the α^2 term corresponds to a "soft-spring" effect In a soft-spring nonlinear system, there is a possibility of an amplitude "jump" in the double-valued region of $\Omega^2 < 1$ As the magnitude of the negative α^2 coefficient increases, the frequency at which a harmonic oscillation can experience a jump decreases, and in principle it can approach zero frequency, i e , a static load jump in amplitude This characteristic can be used to approximate the buckling characteristic of a curved panel The frequency equation may be written $\Omega^2 = 1 + \alpha^2\Omega_2{}^2$, $\Omega_2{}^2 < 0$

If the static buckling is taken to be defined by the condition that α is large enough to cause a jump in amplitude at zero frequency (but $\alpha \ll 1$), then the expression

$$\alpha^2 = -1/\Omega_2{}^2 \qquad (40)$$

specifies the deflection required to cause buckling Note that buckling here has two different meanings, which depend upon whether the initial state is undeflected or buckled equilibrium For Eq (40) to yield a sensible answer, these conditions must obtain $\Omega_2{}^2 < 0$, and $-(1/\Omega_2{}^2)^{1/2} \ll 1$ Typically, this means that $\epsilon \gg 1$, and large ϵ corresponds to a nearly buckled stress state, which is in agreement with the definition of ϵ

There are some interesting differences between Eqs (37a) and (37b) For instance, the maximum negative deflection of buckled equilibrium positions is limited by these conditions:

$$0 > \psi_0 \geq -\tfrac{9}{2} \qquad\qquad 0 > \psi_0 \geq -(1/\chi_{min}) = -\tfrac{7}{18}(4/\pi)^4 \approx -1$$

Table 1 Amplitude-frequency relations, (development A)

ϵ	Undeflected equilibrium frequency	Buckled equilibrium frequency
1	$\Omega^2 = 1 - \frac{2}{3}\alpha^2 - q/\alpha$	$\Omega^2 = 1 - \alpha^2 - q/\alpha$
$\frac{3}{2}$	$\Omega^2 = 1 - \frac{13}{8}\alpha^2 - q/\alpha$	$\Omega^2 = 1 - 1\,591\alpha^2 - q/4\,533\alpha$
2	$\Omega^2 = 1 - 3\alpha^2 - q/\alpha$	$\Omega^2 = 1 - 2\,281\alpha^2 - q/5\,84\alpha$
4	$\Omega^2 = 1 - 12\,16\alpha^2 - q/\alpha$	$\Omega^2 = 1 - 3\,532\alpha^2 - q/14\,88\alpha$

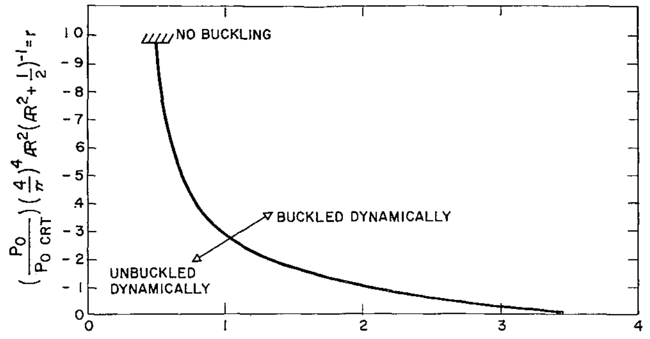

Fig 7 Shock response prediction of buckling for system A

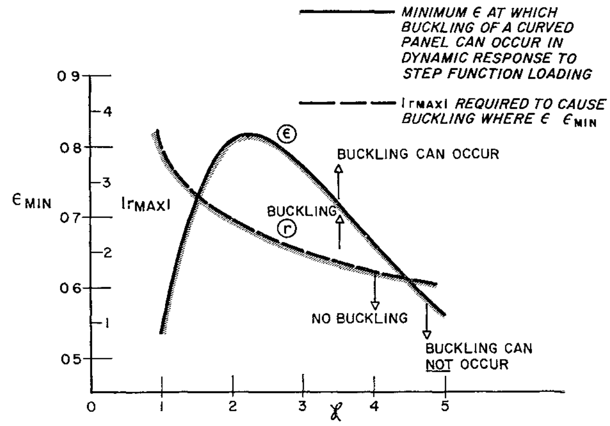

Fig 8 Shock response prediction of buckling for system B

This is a really striking difference in prediction of buckled equilibrium position and may provide a means of evaluating which of these two analyses most nearly represents the physical phenomena under study The sharpness of the tuning at the buckled equilibrium is much less emphasized in the B development and appears as a strong function of aspect ratio through χ, whereas it is a weak function of aspect ratio in the A development

Conclusions

The Galerkin method has been used in two different ways to represent large-amplitude vibrations of a curved panel One representation is identical with that obtained by Reissner[1] in a different way This agreement increases confidence in the accuracy of the Galerkin method for the shallow-shell problem

It is not possible on the basis of the analysis presented here to determine whether system A or system B is definitely superior to the other But examination of the two has led the author to believe that system B is superior to system A in presenting the detailed features of the problem In addition, system B introduces features not appearing in system A although it adds only moderately to the complexity of the analysis However, since system A is similar to system B but contains one parameter fewer (no χ), it is more convenient for illustrating the kinds of effects to be encountered and is heavily relied upon for that purpose in this work

The use of perturbation techniques to represent the periods of both free and harmonically excited vibration yields much information with little effort: Comparison of the perturbation frequencies with the exact frequencies shows that the perturbation method yields good results easily if analysis can be limited to $\alpha \ll 1$ If this restriction cannot be allowed, then the more laborious exact (elliptic integral) solution must be undertaken

Transient response to both delta function and step functions has been treated by simple methods with this conclusion: For step functions, the shock response methods of Fung and Barton make it possible to predict dynamic buckling

Steady motions near the buckled equilibrium position have some features not exhibited by motions near the undeflected equilibrium position In particular, motions are more sharply tuned at the buckled position than at the undeflected position This analysis also reveals the strong influence of initial stresses

Appendix: The Perturbation Solution for Harmonically Excited Vibrations

Consider a nonlinear differential equation that includes both of those developed in this paper, i e ,

$$\psi_{\tau\tau} + \Omega_L^2[\psi + \eta(\psi^2 + \mu\psi^3)] = Q_0 \cos\gamma\tau \qquad \text{(A1)}$$

With the definitions $q = Q_0/\Omega_L^2$, and $\Omega^2 = 1/\Omega_L^2$, Eq (A1) becomes

$$\Omega^2\psi_{\tau\tau} + \psi + \eta(\psi^2 + \mu\psi^3) = q \cos\gamma\tau \qquad \text{(A2)}$$

It is desired to study the behavior of solutions to Eq (A2) which are periodic in form and which have frequencies near to that of the associated linear system Consequently, solutions are sought which are characterized by

$$\begin{aligned} \Omega^2 &= 1 + \alpha\Omega_1^2 + \alpha^2\Omega_2^2 + \\ \psi &= \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + \alpha^3\psi_3 + \\ q &= q_0 + \alpha q_1 + \alpha^2 q_2 + \alpha^3 q_3 + \alpha^4 q_4 + \end{aligned} \qquad \text{(A3)}$$

and $\psi_0 =$ const, with initial conditions

$$\psi_{(\tau=0)} = \psi_0 + \alpha \qquad \psi_\tau(\tau = 0) = 0 \qquad \text{(A4)}$$

In this development, α is a small parameter defined by

$$\alpha = (4n/\pi)^2(h/a)(A_0/h) \qquad \alpha \ll 1 \qquad \text{(A5)}$$

Introducing Eqs (A3) into Eq (A2) and collecting terms according to powers of α yields

$$\alpha^0 \rightarrow \psi_0[1 + \eta(\psi_0 + \mu\psi_0^2)] = q_0 \cos\gamma\tau \qquad \text{(A6)}$$

$$\alpha^1 \rightarrow \psi_{1\tau\tau} + \gamma^2\psi_1 = q_1 \cos\gamma\tau \qquad \text{(A7)}$$

where

$$\gamma^2 = 1 + \eta(2\psi_0 + 3\mu\psi_0^2)$$

$$\alpha^2 \rightarrow \psi_{2\tau\tau} + \gamma^2\psi_2 = q_2 \cos\gamma\tau - \Omega_1^2\psi_{1\tau\tau} - \eta\psi_1^2(1 + 3\mu\psi_0) \qquad \text{(A8)}$$

$$\alpha^3 \rightarrow \psi_{3\tau\tau} + \gamma^2\psi_3 = q_3 \cos\gamma\tau - \Omega_1^2\psi_{2\tau\tau} - \Omega_2^2\psi_{1\tau\tau} - \eta\psi_1[2\psi_2 + \mu(6\psi_0\psi_2 + \psi_1)] \qquad \text{(A9)}$$

If $q_0 = 0$, then the roots of Eq (A6) represent the three singular points in the phase plane This is a very desirable arrangement, since the stable equilibrium points of the undeflected and buckled states associated with Eq (A6) are: 1) if $[1 - 4(\mu/\eta)] < 0$, no buckling is possible, and the static equilibrium point is where $\psi_0 = 0$; and 2) if $[1 - 4(\psi/\eta)] > 0$, buckling is possible, the static equilibrium point is where $(\psi_0)_1 = 0$, and the buckled equilibrium point is where $(\psi_0)_3 = -(1/2\mu)[1 + (1 - 4\mu/\eta)^{1/2}]$

If $q_1 \neq 0$, a term of the form $\tau \cos\gamma\tau$ will appear in the solution to Eq (A7) This is contrary to the requirement that the solution be periodic; therefore $q_1 = 0$ Then the solution to Eq (A7), imposing the initial condition, is

$$\psi_1 = \cos\gamma\tau \qquad \text{(A10)}$$

When Eq (A10) is introduced into Eq (A8), the prohibition against secular terms again requires that the coefficient of the $\cos\gamma\tau$ term vanish The result is that

$$\gamma^2\Omega_1^2 = q_2 \tag{A11}$$

and the differential equation for ψ_2 becomes

$$\psi_{2\tau\tau} + \gamma^2\psi_2 = -(\delta/2)(1 + \cos 2\gamma\tau) \tag{A12}$$

with

$$\delta = \eta(1 + 3\mu\psi_0)$$

The frequency equation is

$$\Omega^2 = 1 - \alpha q_2 = 1 - q/\alpha \tag{A13}$$

which is just the result obtained in the linear analysis A more interesting result is obtained if $q_2 = 0$, and the forcing function is introduced at the same point that the nonlinear effects enter the frequency correction formula

The solution to Eq (A12) is

$$\psi_2 = \phi_0 + \phi_1 \cos\gamma\tau + \phi_2 \cos 2\gamma\tau \tag{A14}$$

where

$$\phi_0 = -\delta/2$$

$$\phi_1 = \delta/2[1 + (1 - 4\gamma^2)^{-1}]$$

$$\phi_2 = -\delta/2[1 - 4\gamma^2]^{-1}$$

When Eq (A14) is introduced into Eq (A9) and the coefficient of $\cos\gamma\tau$ is set equal to zero, the expression for Ω_2^2 is

$$\Omega_2^2 = \frac{\eta}{\gamma^2}\left[3\mu\left(\frac{1}{4} - \psi_0\delta\right) - \left(\frac{3 - 8\gamma^2}{2 - 8\gamma^2}\right)\delta\right] - \frac{q_3}{\gamma^2} \tag{A15}$$

The corresponding frequency expression is

$$\Omega^2 = 1 + \alpha^2\frac{\eta}{\gamma^2}\left[3\mu\left(\frac{1}{4} - \psi_0\delta\right) - \left(\frac{3 - 8\gamma^2}{2 - 8\gamma^2}\right)\delta\right] - \frac{q}{\alpha\gamma^2} \tag{A16}$$

The particular case associated with zero-deflection static equilibrium yields

$$\Omega^2 = 1 + \alpha^2(\epsilon/6)(1 - 5\epsilon) - q/\alpha \tag{A17a}$$

$$\Omega^2 = 1 + \alpha^2(7\epsilon/9)[(3\chi/4) - (35\epsilon/54)] - q/\alpha \tag{A17b}$$

The free-vibration results are obtained by setting $q = 0$

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